3P1-2

Proceedings of Symposium on Ultrasonic Electronics, Vol. 30 (2009) pp. 425-426
18-20 November, 2009

Perfectly Matched Layers in the Cylindrical and Spherical Coordinates for Elastic Waves in Solids

円筒座標ならびに球座標における弾性体の完全整合層

Takao Shimada[1‡] and Koji Hasegawa[2] ([1,2]Graduate School. Eng., Muroran Inst. Tech.)
嶋田賢男 [1‡]，長谷川弘治 [2] （[1,2] 室蘭工大大学院 工学研究科）

1. Introduction

The perfectly matched layer (PML) is one of popular absorbing boundary conditions for truncating the computational domain of open regions without reflection of oblique incident waves. In 1994, Berenger invented a PML for electromagnetic waves in the finite difference time domain (FD-TD) method by a spliting field method.[1)] Because fields in Berenger's PML do not satisfy the Maxwell's equations, two concepts have been introduced for implementation in the finite element method (FEM) of electromagnetic wave problems: the analytic continuation or the complex coordinate stretching[2,3)] and anisotropic PMLs.[4)] Nowadays PMLs for electromagnetic waves are widely used in the FD-TD method and the FEM.

Extension of PMLs to elastic waves in isotropic solids in the Cartesian coordinate first appeared in 1996.[5,6)] In the cylindrical and spherical coordinates, PMLs were presented by using spliting field method in isotropic solids in 1999[7)] and by using analytic continuation in anisotropic solids in 2002.[8)] Recently validity and usefulness of PMLs derived from the analytic continuation in piezoelectric materials was demonstrated.[9-11)]

Although the analytic continuation is powerful tool for derivation of PMLs in the frequency domain, two questions are left: why the particle displacements in the complex coordinate may be identical to those in the real coordinate?; why must we multiply the stress tensors by the Jacobian of the coordinate transformation?

For replying to the questions, we derived PMLs for elastic waves in the Cartesian coordinates from the differential form on manifolds and revealed that the components of stress tensors and the particle displacement vectors in the analytic continuation are not transformed to the real space.[12)]

In this paper, we examine derivation of PMLs in the cylindrical and spherical coordinates from the differential form on manifolds. Our results show that the rule for determing PML parameters in the Cartesian coordinate holds in the cylindrical and spherical coordinates.

1. s2092003@mmm.muroran-it.ac.jp,
2. khasegaw@mmm.muroran-it.ac.jp

2. Differential Form

Particle displacements $\vec{u}$, density of momentums $\vec{P}$, stress tensors $\overline{\overline{T}}$ and displacement gradient tensors $\overline{\overline{F}}$ are given as follows:

$$\vec{u} = u^i \frac{\partial}{\partial x^i} \quad (1),$$

$$\vec{P} = \frac{1}{3!} P^i_{\alpha\beta\gamma} \frac{\partial}{\partial x^i} \otimes dx^\alpha \wedge dx^\beta \wedge dx^\gamma \quad (2),$$

$$\overline{\overline{T}} = \frac{1}{2} T^i_{\alpha\beta} \frac{\partial}{\partial x^i} \otimes dx^\alpha \wedge dx^\beta \quad (3),$$

and

$$\overline{\overline{F}} = F^i_\alpha \frac{\partial}{\partial x^i} \otimes dx^\alpha \quad (4),$$

where $\partial / \partial x^i$ and $dx^\alpha, dx^\beta, dx^\gamma$ are contravariant and covariant basis vectors, $\otimes$ and $\wedge$ represent the tensor product and the cross product, respectively. Newton's equation of motion is $d\overline{\overline{T}} = \partial \vec{P} / \partial t$ where d is the exterior differential operator. Changing the coordinate gives relations of tensor components: for a tensor with tensor type of contravariant of rank 1 and covariant of rank q $V = V^i_{X\alpha_1\ldots\alpha_q} \frac{\partial}{\partial X^i} \otimes dX^{\alpha_1} \wedge \ldots \wedge dX^{\alpha_q} = V^k_{x\beta_1\ldots\beta_q} \otimes dx^{\beta_1} \wedge \ldots \wedge dx^{\beta_q}$, the relation of tensor components is

$$V^i_{X\alpha_1\ldots\alpha_q} = \frac{\partial X^i}{\partial x^k} \frac{\partial x^{\beta_1}}{\partial X^{\alpha_1}} \cdots \frac{\partial x^{\beta q}}{\partial X^{\alpha q}} V^k_{x\beta_1\ldots\beta_q}. \quad (5)$$

Using the complex coordinate stretching[2,3,8)] given by $X^i = \int^{x^i} s_{i\mathrm{R}}(\tau) + \mathrm{j} s_{i\mathrm{I}}(\tau) d\tau$ with the two real functions $s_{i\mathrm{R}}(\tau)$ and $s_{i\mathrm{I}}(\tau)$, we have the relations

$$V^i_{X\alpha_i\ldots\alpha_q} = s_i(x^i) \left[s_{\alpha_1}(x^{\alpha_1}) \ldots s_{\alpha_q}(x^{\alpha_q}) \right]^{-1} V^i_{x\alpha_1\ldots\alpha_q}. \quad (6)$$

Here, $s_i(x^i) = s_{i\mathrm{R}}(x^i) + \mathrm{j} s_{i\mathrm{I}}(x^i)$ and j is the imaginary unit.

3. PMLs in the Cylindrical and Spherical Coordinates

In the complex coordinate stretching, we consider that the real coordinate (x^0, x^1, x^2) is $(\rho, \theta, \bar{z})$ and (r, θ, ϕ) for the cylindrical and the spherical coordinates, respectively. Assuming that the same constitutive equations in the real cylindrical and spherical coordinates exist in the complex coordinate, we have

$$\vec{P}^c = \rho\, \partial \vec{u}^c / \partial t \tag{7}$$

and

$$T_{ij}^c = C_{ijkl} S_{kl}^c = C_{ijkl}(F_{kl}^c + F_{lk}^c)/2 = C_{ijkl} F_{kl}^c \,. \tag{8}$$

Here, the superscript c denotes the value in the complex coordinate and the mass density ρ and the stiffness C_{ijkl} are the values corresponding to original material parameters of its PML in the cylindrical and spherical coordinates. Using eq. (6) to eqs.(1)-(4), we have

$$\vec{P}^c = \frac{1}{s_0 s_1 s_2} \overset{=}{s}_c \cdot \vec{P} \,, \tag{9}$$

$$T_{ij}^c = \frac{s_i s_j}{s_0 s_1 s_2} T_{ij} \text{ (no summation)}, \tag{10}$$

and

$$F_{ij}^c = \frac{s_i}{s_j} F_{ij} \text{ (no summation)}, \tag{11}$$

where $\overset{=}{s}_c = \hat{x}^i \hat{x}^i s_i(x^i)$. The quotient rule and eqs. (7)-(11) yield PML material constants: the mass density ρ^{PML} is

$$\rho^{\mathrm{PML}} = s_0 s_1 s_2 \rho \tag{12}$$

and the stiffness C_{ijkl}^{PML} is

$$C_{ijkl}^{\mathrm{PML}} = \frac{s_0 s_1 s_2 s_k}{s_i s_j s_l} C_{ijkl} \text{ (no summation)}. \tag{13}$$

Eqs. (12) and (13) show that PML parameters for elastic waves in solids in the cylindrical and spherical coordinates may be calculated by the same procedure in the Cartesian coordinates.

4. Comparison with PML Material Constants Derived From Differential Forms and the Analytic Continuation

By the analytic continuation, Zheng and Huang[8] derived the mass density and stiffness of PML in the cylindrical and spherical coordinates: $\rho^{\mathrm{PMLA}} = s_0 s_1 s_2 \rho$ and $C_{ijkl}^{\mathrm{PMLA}} = \dfrac{s_0 s_1 s_2}{s_j s_l} C_{ijkl}$. The mass density agree with our result, eq.(12), because multiplying the stress tensors by the Jacobian of the coordinate transformation, $s_0 s_1 s_2$, adjusts the mass density. We note that the form of eq.(12) is also derived from eq.(6) with the tensor type of mass density being covariant of rank 3, i.e. 3-form. The stiffness C_{ijkl}^{PMLA} is different from eq.(13) because in the analytic continuation, the manipulation of the coordinate transformation corresponding to the part of stress tensor and the particle displacement vector, contravariant of rank 1, is excluded .

5. Conclusions

PMLs in the cylindrical and spherical coordinates for elastic waves in solids were derived from differential forms on manifolds. Our results show that PML parameters for elastic waves in solids may be determined by the same procedure in the Cartesian coordinates. However this rule have been found out for PML material constants derived from the analytic continuation in the cylindrical and spherical coordinates by Zheng and Huang,[8] our derivation based on differential form states that this rule may holds for PML parameters in any orthogonal coordinate system.

Numerical examples for elastic waves in solids of FEM analyses in the frequency domain will be presented elsewhere.

References

1. J.P. Berenger: J. Comput. Phys. **114** (1994) 185.
2. W.C.Chew and W.H. Weedon: Microwave Opt. Tech. Lett. **7** (1994) 599.
3. F.L.Teixeira and W.C.Chew: Int. J. Numer. Model. **13** (2000) 441.
4. Z.S.Sacks, D.M.Kingsland, R. Lee and J.F.Lee: IEEE Trans. Antennas Propag. **43** (1995) 1460.
5. F.D.Hastings, J.B.Schneider and S.L.Broschat: J Acoust. Soc. Am. **100** (1996) 3061.
6. W.C.Chew and Q.H.Liu: J. Comput. Acoust. **4** (1996) 341.
7. Q.H.Liu:J. Acoust. Soc. Am. **105** (1999) 2075.
8. Y.Zheng and X.Huang: MIT Earth Res. Lab. Ind. Consor. Rep. (2002).
9. S.Ballandras,D.Gachon, J.Masson and W.Daniau: Proc. Int. Freq. Contr. Symp., 2007, p.729.
10. M.Mayer, S.Zaglmayr, K.Wagner and J. Schöberl: Proc. IEEE Ultrason. Symp., 2007, p.702.
11. Y.Li, O.B.Matar, V.Preobrazhensky and P. Pernod: Proc. IEEE Ultrason. Symp., 2008, p.1568.
12. K.Hasegawa and T.Shimada: to be published in IEICE Tech. Rep. US09(2009)[in Japanese].