3P-4

Proceedings of Symposium on Ultrasonic Electronics, Vol. 31 (2010) pp. 463-464
6-8 December, 2010

# Discretized Wavenumber Analysis of Reflection Powers of Elastic Waves from a Perfectly Matched Layer in the Finite Element Method

有限要素離散化による完全整合層からの弾性波反射について

Takao Shimada[1†] and Koji Hasegawa[2] ([1,2]Grad. School of Eng., Muroran Inst. Tech.)
嶋田 賢男[1‡], 長谷川 弘治[2] ([1,2]室蘭工大大学院 工学研究科)

## 1. Introduction

Perfectly matched layers[1-5)] (PMLs) for elastic waves in solids are among the popular absorbing boundary conditions for trancating the computational domain of open regions. Mathematical models of PMLs, which are given by differential equations and boundary conditions, are exactly perfect matching medium. In numerical models, however, discretizing PMLs changes phase velocities of propagating waves and generates reflection waves from the PML region.[5)] Furthermore, approximation of infinite regions with finite thick layers also generates reflection waves from the PML terminal.[5)] Estimating matching performance and optimizing parameteres of PMLs are required before solving problems.

In this paper, the dependence of PML performance on attenuation parameters of finite element (FE) models in the frequency range is presented. We confirm that numerical results of FE-models of PMLs for elastic waves may be predicted by discretized wavenumber analysis.

## 2. Basic equations and Numerical procedure

We consider plane elastic waves propagating in a half infinite isotropic solid attached with its PML backed with a vacuum as shown in **Fig.1**. Here $\theta$ is the incident angle, $L$ is thickness of the PML, $\vec{k_i}$ and $\vec{k_m}$ ($m$=0, 1, 2) are wave vectors of the incident wave and reflected wave, respectively. When the stiffness component of an isotropic solid $C_{ijkl}$ is given by $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ where $\lambda$ and $\mu$ are Lame constants and $\delta_{ij}$ is Kronecker's delta, the stiffness components of its PML $C_{ijkl}^{\mathrm{PML}}$ is:

$$C_{ijkl}^{\mathrm{PML}} = [\frac{\lambda}{s_i^2}\delta_{ij}\delta_{kl} + \frac{\mu}{s_j^2}\delta_{ik}\delta_{jl} + \frac{\mu}{s_i^2}\delta_{il}\delta_{jk}]s_0 s_1 s_2 \,.^{5)}$$

Here $s_i\,(i = 0,1,2)$ is a coordinate stretching factor of $x_i$ direction.[2)] The mass density of the PML $\rho^{\mathrm{PML}}$ is given by $\rho^{\mathrm{PML}} = s_0 s_1 s_2 \rho$.

For examining absorbing performance of PMLs in the $x_2$ direction, taking assumptions of uniform field distributions along $x_0$-direction and no variation of fields along $x_1$-direction, we have a differential equation in one variable $x_2$. In this case, we may choose the coordinate stretching factor as follows:

$$s_2 = 1 - j s_{2I}(x_2), \tag{3}$$

$$s_0 = s_1 = 1 \tag{4}$$

where $j$ is the imaginary unit.

In the PML, the differential equation derived from Newton's equation of motion and constitutive equation is:

$$C_{ijkl}^{\mathrm{PML}} \frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial x_l}\right) = \rho^{\mathrm{PML}} \frac{\partial^2 u_i}{\partial t^2} \tag{5}$$

where $u_i$ is the component of the particle displacement in the $x_i$ direction.

In the half isotropic solid, the field distribution, components of the particle displacement and stress, can be expressed by superposition of incident and reflected plane waves:

$$\vec{u}(\vec{r}) = \sum_n R_n \vec{u}_n e^{-j\vec{k}_n\cdot\vec{r}} + \vec{u}_i e^{-j\vec{k}_i\cdot\vec{r}} \,. \tag{6}$$

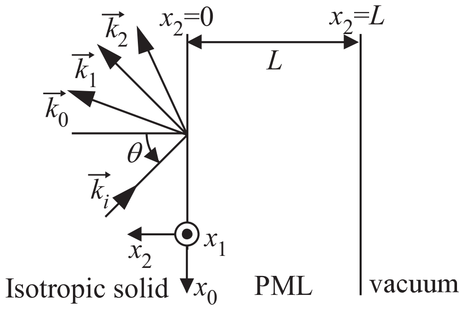


Fig. 1 Reflection on the plane boundary between an isotropic solid and its PML.

---
2.khasegaw@mmm.muroran-it.ac.jp

Where $\vec{k}_n$ and $\vec{u}_n$ is the wave vector and the particle displacement vector that are given by the solutions of Christoffel equation for the isotropic solid.

Boundary conditions at the interface of isotropic solid and PML, $x_2$=0, are the nonslip condition and the continuous condition of the normal component of the stress:

$$\vec{u}(+0) = \bar{\bar{s}} \cdot \vec{u}(-0) \quad (7)$$

$$T_{2j}(+0) = \frac{s_2 s_j}{s_0 s_1 s_2} T_{2j}(-0) \quad (8)$$

where

$$\bar{\bar{s}} = \hat{x}_i \hat{x}_i s_i . \quad (9)$$

At the terminal of PML, $x_2$=$L$, the boundary condition is:

$$\frac{s_2 s_j}{s_0 s_1 s_2} T_{2j}(L) = 0. \quad (10)$$

Using the boundary condition and following relations at $x_2$=0,

$$\vec{u}(+0) = [u_{mn}]\{R_0 \ R_1 \ R_2\}^T + \vec{u}(+0) \quad (11)$$

$$[u_{mn}][k_{mn}]^{-1} \left.\frac{\partial \vec{u}}{\partial x_2}\right|_{x_2=+0} - \vec{u}(+0) = -\left([\mathrm{I}] + (j\vec{k}_i \cdot \hat{x}_2)[u_{mn}][k_{mn}]^{-1}\right)\vec{u}_i(+0) \quad (12)$$

where $k_{mn} = (-j\vec{k}_n \cdot \hat{x}_2)u_{mn}$ and subscripts $m$ and $n$ indicate $x_0$-, $x_1$-, $x_2$-component of the coordinate and P-, SV- and SH-wave, respectively, we get a Robin condition. Hence applying finite element procedure to the PML region only, we can determine value of the particle displacement $\vec{u}(-0)$, and compute the reflection coefficient $R_n$ by eqs. (7), (8), (11), (12).

## 3. Reflection coefficients calculated from discretized wavenumber

Numerical dispersions due to the finite line element approximation of the propagating fields in the PML, as shown in **Fig.2**, change the wavenumbers predicted by differential equations to discretized wavenumbers[6] as follows:

$$\tilde{k} = \begin{cases} \dfrac{1}{h}\cos^{-1}\left[\dfrac{6-2(k_L s_2 h)^2}{6+(k_L s_2 h)^2}\right] & \text{P-wave} \\ \dfrac{1}{h}\cos^{-1}\left[\dfrac{6-2(k_S s_2 h)^2}{6+(k_S s_2 h)^2}\right] & \text{SV-wave} \end{cases}$$

where $k_L$ and $k_S$ are the intrinsic wavenumber of P- and SV-wave in the isotropic region.

Identification of the structure shown Fig.1 as a layered structure, we can compute the reflection coefficient by the discretized wavenumber.

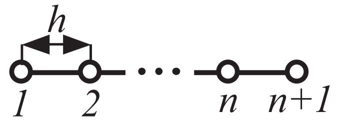


Fig.2 Line element with ($n$+1)-nodes

## 4. Computed Results

**Fig.3** shows the computed results of the reflection coefficient dependence on $\lambda / h$ for Poisson ratio $\sigma$=0.3, $s_{2I}(x_2) = 0.1$, $n$=1, $\theta = 0^\circ$ and normalized thickness $k_s L = 24\pi$. Here $\lambda$ is the wavelength of incident SV-wave in the isotropic region. We confirm that the results computed by discretized wavenumber (○,□) agree with those by FE method (—,---).

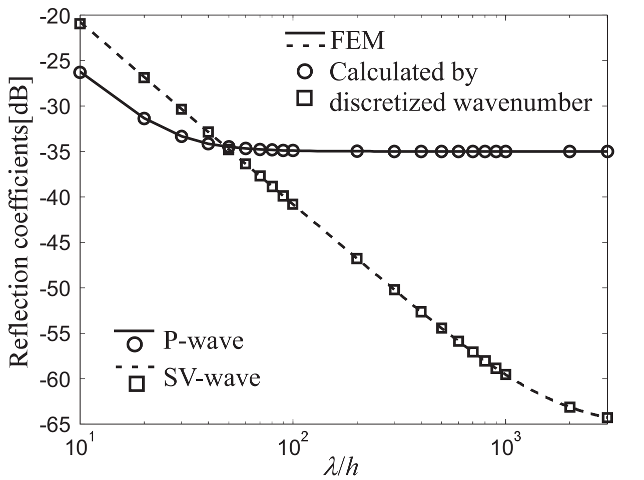


Fig. 3 Reflection coefficients on the plane boundary between an isotropic solids and its PML.


**References**

1.J.P.Berenger: J. Comput. Phys. **114** (1994) 185.

2.W.C.Chew and W.H.Weedon: Microwave Opt. Technol. Lett. **7** (1994).

3.Y.Zheng and X.Huang:MIT Earth res. Lab. Ind. Consortium Rep.(2002).

4 T. Shimada, K. Hasegawa: Jpn. J. Appl. Phys. **49** (2010) 07HB08.

5.T.Shimada, K.Hasegawa: IEICE Trans. **J93-C** (2010) 215.

6.T.Shimada, K.Hasegawa and S.Sato:Keisan Suri Kogaku Ronbunshu **9** (2009) 19 [in japanese].