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Grids in the Finite-Difference Time-Domain Method for Elastic Waves in Solids

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1. Introduction

Finite-difference time-domain method is an attractive tool for modeling propagation and scattering of ultrasonic waves in a solid in the time domain. For an isotropic solid, staggered grids of velocity and stress models with the second-order accuracy in time difference and second- or fourth-order accuracy in spatial difference schemes, (2,2) and (2,4) schemes, are reported.[1,2] In these grids the central difference approximation of time derivatives result in staggered grids of stresses and velocities by half of time interval $\Delta t/2$. In addition for isotropic solids under the condition of the elastic fields being uniform along the y-axis, the nodes of *x*- and *z*-components of velocities are staggered by $\Delta/2$, which is half of the spatial size of a uniform unit cell, along *x*- and *z*-axes because of finite difference approximations of the relation between strains and partial differentiations of particle velocities. These arrangements of the grids of (2,2) and (2,4) schemes show that control volumes corresponding to the law of the momentum conservation are staggered for *x*- and *z*-components in the finite integration technique (FIT).[3] Therefore proper setting of control volumes for each component at the heterogeneous material boundary is required for imposing the boundary condition on the velocities and stresses. In addition, for elastic waves in anisotropic solids, control volumes of grids have *x*- and *z*-components of velocities on each node and each control volume shares one quarter with neighborly volumes. Therefore, proper setting of control volumes becomes complex.

For removing this disadvantage, we present a grid derived from a control volume for all components of the momentum conservation law and the displacement gradient.

2. Differential Form

Particle displacements $\vec{u}$, density of momentums $\vec{P}$, stress tensors $\overline{\overline{T}}$ and displacement gradient tensors $\overline{\overline{F}}$ are given as follows:

$$\vec{u} = u^i \frac{\partial}{\partial x^i} \quad (1),$$

$$\vec{P} = \frac{1}{3!} P^i_{\alpha\beta\gamma} \frac{\partial}{\partial x^i} \otimes dx^\alpha \wedge dx^\beta \wedge dx^\gamma \quad (2),$$

$$\overline{\overline{T}} = \frac{1}{2} T^i_{\alpha\beta} \frac{\partial}{\partial x^i} \otimes dx^\alpha \wedge dx^\beta \quad (3),$$

and

$$\overline{\overline{F}} = F^i_\alpha \frac{\partial}{\partial x^i} \otimes dx^\alpha = d\vec{u} \quad (4),$$

where $\partial / \partial x^i$ and $dx^\alpha, dx^\beta, dx^\gamma$ are contravariant and covariant basis vectors, $\otimes$ and $\wedge$ represent the tensor product and the cross product, respectively. Newton's equation of motion is

$$d\overline{\overline{T}} = \partial \vec{P} / \partial t \quad (5),$$

where d is the exterior differential operator.

3. Spatial Grids in the Cartesian Coordinate

For simplicity we consider an (2,2) scheme in the Cartesian coordinate (x_1, x_2, x_3). Applying central difference approximation with interval Δx_i in the x_i-coordinate to partial derivatives of x_i-components in the eq. (4) and eq. (5), we have two spatial grids as shown in **Fig. 1** and **Fig. 2**, respectively. Here time staggered arrangement is omitted: the components of a displacement gradient tensor in Fig. 1 and the components of the momentum in Fig. 2 should be replaced with its partial time derivatives. Spatial grids of velocity and stress models for propagating elastic waves in solids are constructed from these two grids and the constitutive equations as follows:

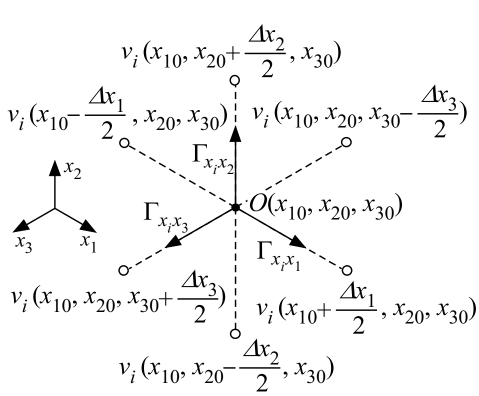

Fig. 1. A spatial grid of x_i-components of the velocity vector and the displacement gradient tensor at the point $O(x_{10}, x_{20}, x_{30})$. Note that time-staggered arrangement is neglected.

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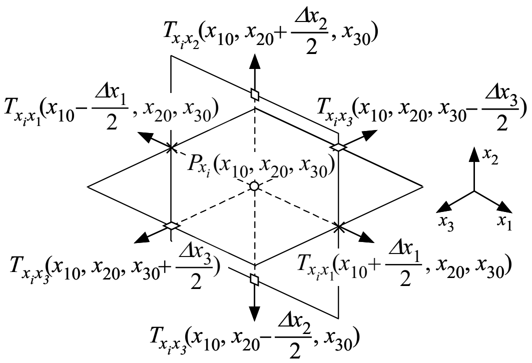

Fig.2. A spatial grid of x_i-components of the momentum vector and the stress tensor at the point O(x_{10}, x_{20}, x_{30}).

$$\vec{P} = \rho\, \partial\vec{u} / \partial t \tag{6}$$

and

$$T_{ij} = C_{ijkl} S_{kl} = C_{ijkl}(F_{kl} + F_{lk})/2 = C_{ijkl} F_{kl}\,. \tag{7}$$

Here ρ and C_{ijkl} are the mass density and the stiffness respectively.

Note that eq. (5) and FIT with a control volume being a cuboid, $x_{i0} - \frac{\Delta x_i}{2} \le x_i \le x_{i0} + \frac{\Delta x_i}{2}$ for i=1,2,3, give a grid with collocated components of the velocity vector straightforward. Eq. (4) and FIT with a line integral of i-component of the velocity vector v_i along k-coordinate put the ik-component of the displacement gradient tensor F_{ik} on the same node of stress component T_{ik} from Figs. 1 and 2. Hence we can build a new grid with collocated components of the velocity vector. We remark that computing T_{ik} in eq. (7) requires unknown components of $\overline{\overline{F}}$ on the nodes and then these components can be computed by the polynomial interpolation of velocity-values on adjacent nodes. When we use the strain-displacement relation, $S_{kl}=(F_{kl}+F_{lk})/2$, this yields cross-shaped arrangement of velocity components, v_k and v_l, as can be seen from Fig. 1 and may inhibit the development of grids with collocated components of the velocity vector.

4. Numerical Dispersion Relations of Grids for P- and SV-waves in an Isotropic Solid

We consider plane P- and SV-waves propagating in an infinite isotropic solid with Poisson's ratio 0.25. Numerical dispersion relations of new grids of (2,2) and (2,4) schemes with the first- or third-order polynomial interpolation for P-waves with wave-vector $\vec{k}$ are as follows: the (2,2) scheme with the first-order interpolation

$$(\lambda + 2\mu)(s_x^2 + s_z^2) - (\lambda + \mu) s_x^2 s_z^2 - \rho \left(\frac{\Delta}{\Delta t}\right)^2 s_\omega^2 = 0,$$

the (2,2) scheme with the third-order interpolation

$$\frac{\lambda+3\mu}{2}(s_x^2 + s_z^2) - \rho\left(\frac{\Delta}{\Delta t}\right)^2 s_\omega^2 + \frac{\lambda+\mu}{2}\Big[(s_x^2+s_z^2)^2$$
$$-4 s_x^2 s_z^2 \big(1-(s_x^2+2)^2(s_z^2+2)^2(1-s_x^2)(1-s_z^2)/16\big)\Big]^{1/2} = 0,$$

the (2,4) scheme with the first-order interpolation

$$\frac{\lambda+3\mu}{2}\big(s_x^2(s_x^2+6)^2 + s_z^2(s_z^2+6)^2\big) - 36\rho\left(\frac{\Delta}{\Delta t}\right)^2 s_\omega^2 + \frac{\lambda+\mu}{2}\Big[\big(s_x^2(s_x^2+6)^2$$
$$+ s_z^2(s_z^2+6)^2\big)^2 - 4 s_x^2 s_z^2 (s_x^2+6)^2 (s_z^2+6)^2 (s_x^2 + s_z^2 - s_x^2 s_z^2)\Big]^{1/2} = 0,$$

and the (2,4) with the third-order interpolation

$$\frac{\lambda+3\mu}{2}\big(s_x^2(s_x^2+6)^2 + s_z^2(s_z^2+6)^2\big) - 36\rho\left(\frac{\Delta}{\Delta t}\right)^2 s_\omega^2$$
$$+\frac{\lambda+\mu}{2}\Big[\big(s_x^2(s_x^2+6)^2 + s_z^2(s_z^2+6)^2\big)^2$$
$$-4 s_x^2 s_z^2 (s_x^2+6)^2 (s_z^2+6)^2 \big(1-(s_x^2+2)^2(s_z^2+2)^2(1-s_x^2)(1-s_z^2)/16\big)\Big]^{1/2} = 0.$$

Where λ and μ are Lame' constants, and

$s_x = \sin\left(\vec{k}\cdot\hat{x}_1 \frac{\Delta}{2}\right)$, $s_z = \sin\left(\vec{k}\cdot\hat{x}_3 \frac{\Delta}{2}\right)$, and $s_\omega = \sin\left(\omega \frac{\Delta t}{2}\right)$.

We confirm that higher order interpolations of spatial derivative or unknown $\overline{\overline{F}}$ components add compensatory terms to the dispersion relations. **Fig. 3** shows computed numerical velocities of P-wave V_{NP} and SV-wave V_{NS} with $V_p \Delta t / \Delta = 0.5$ and $\lambda_s / \Delta = 10$. Here $V_P = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$, $V_S = \sqrt{\dfrac{\mu}{\rho}}$ and λ_s is the wavelength of the SV-wave. Accuracy of computed velocity of new grid of (2,2) or (2,4) scheme with third-order interpolation is comparable with the conventional scheme.

References

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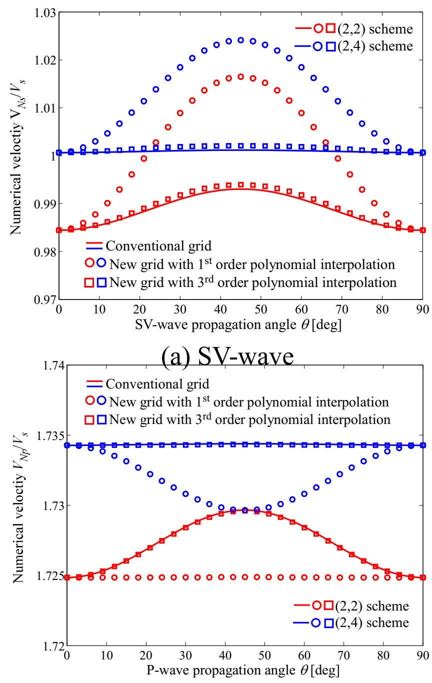

(a) SV-wave

(b) P-wave

Fig. 3. Computed numerical dispersions for P- and SV-waves in an isotropic solid with Poisson's ratio 0.25