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Acoustic resonance of two-dimensional elastic sheet studied by Airy stress function

Airy の応力関数を用いた二次元線形弾性体の共鳴振動理論

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1. Introduction

It is well known that elastic constants C_{ij} plays an important role in materials science and condensed matter physics. Up to date, several methods have been proposed to determine the C_{ij} tensor from experiment such as tensile test, ultrasound pulse-echo method, nano-indentation method, etc. To the best of the authors knowledge, resonant ultrasound spectroscopy (RUS)[1,2] is the state of the art technique because (i) it determines a complete set of C_{ij} tensor from one single crystal sample and (ii) measurement accuracy is sufficiently high; generally, inaccuray is less than 1 %. Because of these reasons that the RUS method has been applied to various kinds of materials. In the theory of RUS, however, there exists an mathematical issue that should be addressed. Since the pioneering work by Ritz, resonant state has been characterised by a stationary condition of the action integral I and which is solved numerically by a direct method. Here, I is defined as a functional of deformation function u_i. It is therefore easy to handle a fixed boundary condition (or Dirichlet type boundary condition). However, free vibration requires stress-free boundary condition (or natural boundary condition) and this condition is satisfied if only if the order of basis function becomes infinity. In order to avoid the mathematical difficulty, in the present study, we propose a new theory for free vibration acoustic resonance of two-dimensional elastic sheet. Our theory uses Airy stress function and satisfies the natural boundary condition explicitly under finite number of basis function.

2. Free Vibration Acoustic Resonance

2-1. The theory of RUS

Let us consider a three-dimensional elastic body defined by $\Omega = \{x_i | -L_i < x_i < L_i, i = 1,2,3\}$ and let u_i, $u_{i,j}$ and $u_{i,t}$ be deformation function, spatial derivative of u_i and time derivative of u_i in Ω. Then, the action integral I can be expressed by the following form.

$$I[u_i] = \int_{t_1}^{t_2} \int_{\Omega} \mathcal{L}(u_{i,t}, u_{i,j})\, dxdt. \qquad (1)$$

Here $\mathcal{L}$ is called the Lagrangian density. Since resonance condition is characterized by $\delta I = 0$, we have the following Euler-Lagrange equation and natural boundary conditions.

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_{i,t}} + \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}}{\partial u_{i,j}} = 0. \qquad (2)$$

$$\left.\frac{\partial \mathcal{L}}{\partial u_{i,t}}\right|_{t=t_1}^{t=t_2} = 0,\ \ \mathcal{L}\Big|_{t=t_1}^{t=t_2} = 0,\ \ \left.\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right|_{x_j=-L_j}^{x_j=L_j} = 0. \qquad (3)$$

Eq. (2) is linear wave equation defined in Ω. The first two identities in Eq. (3) require that the deformation function u_i must be periodic with respect to time. The third condition concerns on the stress-free boundary condition which is mentioned in the previous section. Generally, there is no analytic solution for Eqs. (2) and (3) if Ω has a rectangular parallelepiped shape. Thus, we solve the stationary condition $\delta I = 0$ numerically by a direct method. However, since the action integral I is defined as a functional of deformation u_i, we cannot impose the stress-free boundary condition directly to the basis function. Hence, strictly speaking, the boundary condition cannot be satisfied if the order of basis function is finite.

2-2. The theory of RUS

In order to avoid the mathematical difficulty described above, we formulate free vibration acoustic resonance of a solid by using Airy stress function rather than deformation u_i. To simplify the analysis, for the first approximation, we employ a two-dimensional isotropic elastic sheet model:

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$\Omega = \{x, y | -L_1 < x < L_1, -L_2 < y < L_2\}$. Let φ be Airy stress function defined in Ω. Then, the strain energy of Ω becomes the functional of φ such that

$$E[\varphi] = \int_\Omega \left(\frac{{\varphi_{yy}}^2 + {\varphi_{xx}}^2 + 2{\varphi_{xy}}^2}{4\mu} - \frac{\lambda(\varphi_{yy} + \varphi_{xx})^2}{4\mu(3\lambda + 2\mu)}\right) dV. \quad (4)$$

Here λ and μ are Lame constants. In order to avoid the trivial solution, we impose the following subsidiary condition

$$||\varphi||_{L^2} = \int_\Omega |\varphi|^2 dV = const. \quad (5)$$

Let us express the Lagrange undetermined multiplier by κ. Then, from Eqs. (4) and (5), we obtain the following functional.

$$J[\varphi] = \int_\Omega \left(\frac{{\varphi_{yy}}^2 + {\varphi_{xx}}^2 + 2{\varphi_{xy}}^2}{4\mu} - \frac{\lambda(\varphi_{yy} + \varphi_{xx})^2}{4\mu(3\lambda + 2\mu)} - \kappa|\varphi|^2\right) \mathrm{dV}. \quad (6)$$

The stationary condition $\delta J = 0$ yields minimization of energy functional Eq. (4) under the constraint condition of Eq. (5). To solve the variational problem, we expand the Airy stress function in such a way that

$$\varphi = (x^2 - {L_1}^2)^2 (y^2 - {L_2}^2)^2 \sum_{m=0}^{M} \sum_{n=0}^{N} (\alpha_{m,n} x^m y^n). \quad (7)$$

Note that this stress function satisfies the stress-free boundary condition irrespective to the coefficients $\alpha_{m,n}$. Inserting Eq. (7) into Eq. (6), it ends up with a linear eigenvalue problem: κ and $\alpha_{m,n}$ are determined as eigenvalue and eigenvector, respectively. For numerical analysis, we set $L_1 = 0.9$, $L_2 = 1.3$ and $\lambda = 3\mu = 1$. The order of basis function is set to be $M = N = 5$: the degree of freedom is $\chi = (M+1)(N+1) = 36$. This means that we obtain 36 types of stress function φ and Lagrange undetermined multiplier κ.

3. Results and Discussion

Figure 1 shows two-dimensional stress distribution in Ω calculated from the first three vibration modes. As seen from the figures, σ_{11} and σ_{12} vanish at x = L_1 and -L_1. Similarly, σ_{22} and σ_{12} vanish at y = L_2 and $-L_2$ as required. It should be noted here that the stress distributions plotted in Fig. 1 satisfy the equilibrium equation $\sigma_{ij,j} = 0$ since they are derived from the Airy stress function. In addition, if we set $||\varphi||_{L2} = 1$, then the 36 types of stress functions are orthogonal in a sense that

$$\langle \varphi_i, \varphi_j \rangle = \int_\Omega \varphi_i \varphi_j dV = \delta_{ij}. \quad (8)$$

Furthermore, the eigenvalue κ obtained from the method showed linear relationship with those obtained from the previous method. These results strongly suggest that the present theory provides an appropriate way for describing free vibration acoustic resonance of two-dimensional elastic sheet.

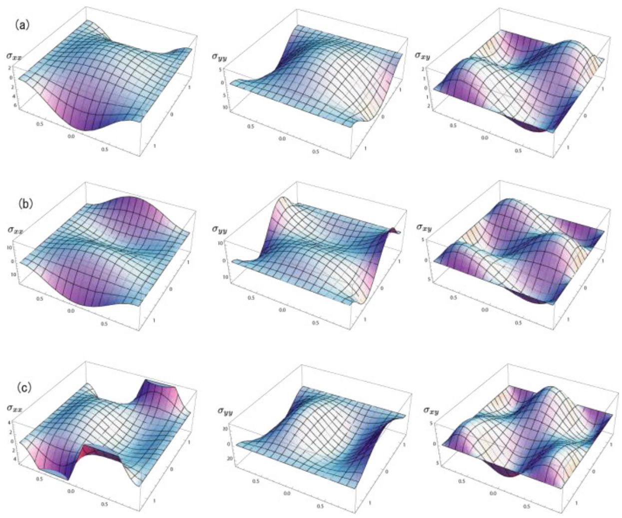

Figure 1. Stress distributions obtained from the first three resonant vibration modes; (a) κ = 39.2673, (b) κ = 98.6992 and (c) κ = 230.018.

4. Conclusions

This study presents new theory for acoustic resonance of two-dimensional elastic sheet. Our formulation is based on Airy stress function φ and the calculus of variation. Numerical analysis revealed that the stress function φ satisfies stress-free boundary condition and equilibrium equation simultaneously. It is also found that the eigenvalue κ showed linear relationship with those obtained from the previous method.

References

1. H. Demarest Jr., J. Acoust. Soc. Am. **49**, 768 (1971).
2. I. Ohno et al, J. Acoust. Soc. Am, **110**, 830 (2001).