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A Finite-Difference Time-Domain Technique for Nonlinear Elastic Media and Its Application to Nonlinear Lamb Wave Propagation

非線形弾性体に対する FDTD 法の定式化とラム波非線形伝搬解析への応用

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1. Introduction

There have recently been a number of studies of elastodynamic problems using the finite-difference time-domain (FDTD) method[1]. FDTD method is first introduced by Virieux[2] as the velocity-stress finite-difference or staggered grid finite-difference formulation. A grid for conventional FDTD is shown in Fig.1(a). This method has several desirable features. In FDTD formulation, Navier's equations are decomposed to a set of first-order partial differential equations with respect to velocity and stress. The staggerd grid finite-difference formulation has practical advantage for its stability and accuracy[3]. Most of previous studies of FDTD simulation, however, assumed linear elastic bodies. Recent research has demonstrated that nonlinear ultrasonic waves can be used to evaluate the material degradation sensitively. In this regard, it is important to gain understanding of the nonlinear response in ultrasonic wave propagation.

In the present study, a formulation to deal with finite amplitude waves based on FDTD method is presented. The kinematic as well as material nonlinearities are considered in this formulation, employing the expression of strain energy by Landau and Lifshitz[4]. The solid is assumed to be isotropic.

Some results of numerical simulation applied to Lamb waves are shown below based on this formulation. The dispersion curves constructed by the numerical results are compared to the analytical ones given by the Rayleigh-Lamb frequency equations. Furthermore, in the situation where a condition[6] of phase matching of fundamental and harmonic Lamb modes holds, cumulative harmonic generation is demonstrated as one of nonlinear effects in Lamb waves.

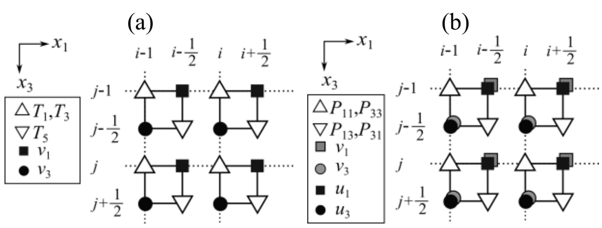

Fig.1 Geometry of (a) the conventional FDTD grid and (b) the FDTD grid for nonlinear simulation

2. Equations of nonlinear elastodynamics and discretization

To deal with finite amplitude waves, two sources of nonlinearity should be taken in account: the material nonlinearity and the kinematic nonlinearity. To include the former, we consider the contribution of the terms in the strain energy density which are cubic in the strains. The strain energy density W is given by

$$\rho_0 W = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} + \frac{1}{6} C_{ijklmn} E_{ij} E_{kl} E_{mn} + \cdots, \tag{1}$$

where ρ_0 is the mass density in the unstressed configuration, E_{ij} is the Lagrangian strain tensor, $E_{ij} = 1/2\ (\partial u_i/\partial X_j + \partial u_j/\partial X_i + \partial u_k/\partial X_i\ \partial u_k/\partial X_j)$ and C_{ijkl} and C_{ijklmn} are the second- and third-order stiffness tensors, respectively. The equations of motion can be written by [5]

$$\rho_0 \frac{\partial v_i}{\partial t} = \frac{\partial P_{ij}}{\partial X_j}, \quad \frac{\partial u_i}{\partial t} = v_i, \tag{2}$$

$$\begin{aligned} P_{ij} &= C_{ijkl} \frac{\partial u_k}{\partial X_l}, \\ &+ \frac{1}{2}\left(C_{ijklmn} + C_{ijln}\delta_{km} + C_{jnkl}\delta_{im} + C_{jlmn}\delta_{ik}\right) \frac{\partial u_k}{\partial X_l} \frac{\partial u_m}{\partial X_n}, \end{aligned} \tag{3}$$

where X_i are the Lagrangian (or material) coordinates, v_i denote the velocity, u_i are the displacement and P_{ij} are the components of a non-symmetric tensor known as the first Piola-Kirchoff stress tensor. In this formulation, P_{ij} need the spacial gradients of the displacement. For this reason, the displacement is calculated by the integral of the velocity. The grid for this formulation is shown in Fig.1(b). Given that the solid is isotropic, the strain energy density W is given by following equation:

$$\begin{aligned} \rho_0 W &= \mu E_{ik}{}^2 + \frac{1}{2} \lambda E_{ll}{}^2 \\ &+ \frac{1}{3} A E_{ik} E_{il} E_{kl} + B E_{ik}{}^2 E_{ll} + \frac{1}{3} C E_{ll}{}^3 + \cdots, \end{aligned} \tag{4}$$

where λ and μ are the Lamé elastic constants and A, B and C are the third-order elastic constants used by Landau and Lifshitz[4]. The stiffness tensors in Eq.(1) and (3) are given by

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2\mu I_{ijkl}, \tag{5}$$

$$C_{ijklmn} = \frac{A}{2}\left(\delta_{ik}I_{jlmn} + \delta_{il}I_{jkmn} + \delta_{jk}I_{ilmn} + \delta_{jl}I_{ikmn}\right) + 2B\left(\delta_{ij}I_{klmn} + \delta_{kl}I_{mnij} + \delta_{mn}I_{ijmn}\right) + 2C\delta_{ij}\delta_{kl}\delta_{mn}, \quad (6)$$

where δ_{ij} is the Kronecker's delta and $I_{ijkl}=(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})/2$.

3. Application to Lamb wave propagation

We show the numerical results obtained by using the above formulation. The simulation model is a 2-dimensional cross section of aluminum plate (2mm thick, 400mm long, c_L=6350m/s, c_T=3130 m/s). The grid size is Δx=0.02mm and the number of the grid points are 20000×100. The incident wave is a sinusoidal wave and is excited from one side of the plate. The simulation was performed by increasing the frequency of the incident wave from 0.5 to 5.0MHz. The results are analyzed in the frequency-wave number (*f-k*) plot obtained by 2-dimensional Fourier Transform. The plotted points in Fig.2 denote the peaks of the *f-k* plot from the numerical simulation and the solid and dashed lines are the theoretical dispersion curves which are solutions of the Rayleigh-Lamb frequency equations. The plotted points generated by the numerical simulation show good agreement with the theoretical curves. This indicates the validity of this formulation.

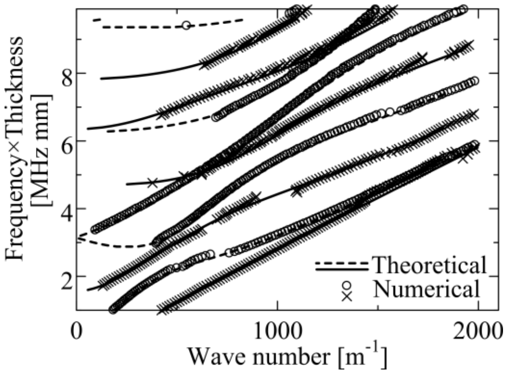

Fig.2 *f-k* peaks of the numerical results and the analytical dispersion curves by the Rayleigh-Lamb equations.

In aluminum plate, a Lamb wave (S_1 mode and 1.80MHz) is expected to generate harmonics in a cumulative manner[6]. We conducted numerical simulations when S_1 mode Lamb wave was excited with the fundamental frequencies of 1.80 MHz. Fig.3 shows the result of the 2-dimensional Fourier transform. Solid and dashed lines in Fig.3 denote the theoretical dispersion curves. In this Figure, we can observe the second harmonic (S_2 mode, 3.60MHz) generated from the fundamental mode. The amplitude ratio of the second harmonic and the fundamental wave was computed and shown in Fig.4 as a function of the propagation distance of the fundamental Lamb wave. The relative amplitude of second harmonic increases linearly as a function of the propagation distance. This shows that the second harmonic is generated in a cumulative manner.

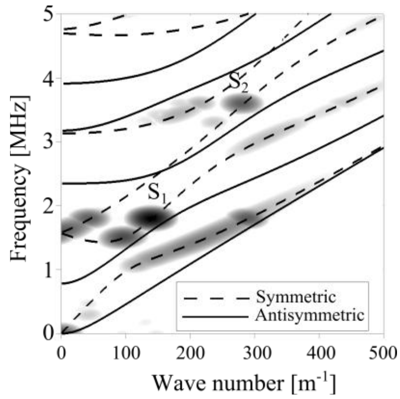

Fig.3 *f-k* distribution by the numerical simulation at the propagation distance of 100 mm.

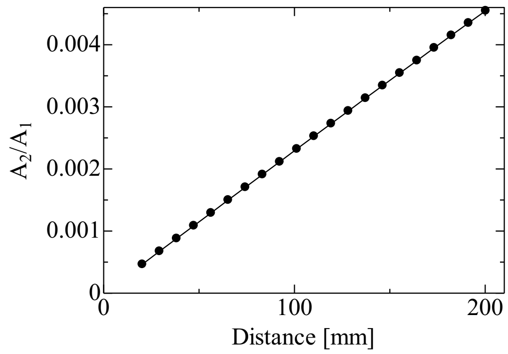

Fig.4 Variation of the relative amplitude of the second harmonics with the propagation distance.

4. Conclusion

A numerical formulation of the FDTD method for finite amplitude ultrasonic waves has been derived. The elastic body is assumed to be isotropic. The kinematic nonlinearity and the material nonlinearity based on the strain energy by Landau and Lifshitz are considered. The results of numerical simulation based on this formulation gave good agreement with the theoretical dispersion curves. This model can also simulate the cumulative second harmonic generation in Lamb waves.

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