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Perfectly Matched Layers of Elastic Wave Propagation in a Piezoelectric Solid in the Frequency Domain

周波数領域における圧電弾性波の完全整合層について

Koji Hasegawa [1†] and Shingo Sato[2] ([1,2]Graduate School. Eng., Muroran Inst. Tech.)
長谷川弘治 [1†]，佐藤　慎悟 [2] （[1,2]室蘭工大大学院 工学研究科）

1. Introduction

A perfectly matched layer (PML) is an absorbing boundary conditions for truncating the computational domain of open regions without reflection of oblique incident waves. In 1994, Berenger invented a PML for electromagnetic waves in the finite difference time domain (FD-TD) method by a spliting field method.[1)] Because fields in Berenger's PML do not satisfy the Maxwell's equations, two concepts have been introduced for implementation in the finite element method (FEM) of electromagnetic wave problems: the analytic continuation or the complex coordinate stretching[2,3)] and anisotropic PMLs.[4)] Nowadays PMLs for electromagnetic waves are widely used in the FD-TD method and the FEM.

Extension of PMLs to elastic waves in isotropic solids in the Cartesian coordinate first appeared in 1996.[5,6)] In the cylindrical and spherical coordinates, PMLs were generated by using spliting field method in isotropic solids in 1999[7)] and by using analytic continuation in anisotropic solids in 2002.[8)] Recently validity and usefulness of PMLs derived from the analytic continuation in piezoelectric solids in the Cartesian coordinates was demonstrated.[9-11)]

From the differential form on manifolds, we have derived PMLs for elastic waves in the Cartesian,[12)] cylindrical and spherical coordinates[13)] and revealed that the contravariant components of stress tensors and the particle displacement vectors in the analytic continuation are not transformed to the real space.[12)] Therefore, the discrepancy in the stiffness constants derived from two methods exists.

In this paper, we examine a derivation of PMLs of elastic wave propagation in piezoelectric solids from the differential form on manifolds. PML material parameters in the orthogonal coordinate systems such as the Cartesian and cyrindrical coordinates are presented. A dispcrepancy in piezoelectric stress constants exists in addition to the stiffness constants appeared in nonpiezoelectric solids in the Cartesian coordinates.[12,13)] The different transformation rules for the contravariant components cause this discrepancy.

2. Differential Form

Particle displacements $\boldsymbol{u}$, densities of momentums $\boldsymbol{P}$, stress tensors $\bar{\bar{T}}$, and displacement gradient tensors $\bar{\bar{F}}$ are given as follows:

$$\boldsymbol{u} = u^i \frac{\partial}{\partial x^i}, \tag{1}$$

$$\boldsymbol{P} = \frac{1}{3!} P^i_{\alpha\beta\gamma} \frac{\partial}{\partial x^i} \otimes dx^\alpha \wedge dx^\beta \wedge dx^\gamma, \tag{2}$$

$$\bar{\bar{T}} = \frac{1}{2} T^i_{\alpha\beta} \frac{\partial}{\partial x^i} \otimes dx^\alpha \wedge dx^\beta, \tag{3}$$

$$\bar{\bar{F}} = \frac{1}{2} F^i_{\alpha} \frac{\partial}{\partial x^i} \otimes dx^\alpha, \tag{4}$$

where $\partial/\partial x^i$ and dx^i ($i,\alpha,\beta,\gamma = 0,1,2$) are contravariant and covariant basis vectors, $\otimes$ and $\wedge$ represent the tensor product and the cross product, respectively. Newton's equation of motion is

$$d\bar{\bar{T}} = \frac{\partial \boldsymbol{P}}{\partial t}, \tag{5}$$

where d is the exterior differential operator.

Using a quasi-static approximation of electromagnetic fields in piezoelectric solids with omitting rotational electric fields and representing electric fields as $E = -d\phi$, we may consider the electric potentials ϕ, the irrotational electric fields $\boldsymbol{E}$, electrical displacements $\boldsymbol{D}$, and the Gauss law in the piezoelectric solids for computing elastic fields coupled with electromagnetic fields.

The electric potentials are scalars, whose tensor type are contravariant and covariant of rank 0. Two vectors $\boldsymbol{E}$, $\boldsymbol{D}$ and the Gauss law are given as follows:

$$\boldsymbol{E} = E_\alpha dx^\alpha, \tag{6}$$

$$\boldsymbol{D} = \frac{1}{2} D_{\alpha\beta} dx^\alpha \wedge dx^\beta, \tag{7}$$

$$d\boldsymbol{D} = 0. \tag{8}$$

Changing the coordinate gives relations of tensor components in the two coordinates: for a tensor V with a tensor type of contravariant of rank 1 and covariant of rank q, $V = V^i_{X\alpha_1\alpha_2\cdots\alpha_q} \frac{\partial}{\partial X^i} \otimes dX^{\alpha_1} \wedge \cdots \wedge dX^{\alpha_q} = V^k_{x\beta_1\beta_2\cdots\beta_q} \frac{\partial}{\partial x^k} \otimes dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_q}$, the relation of tensor components is

$$V^i_{X\alpha_1\alpha_2\cdots\alpha_q} = \frac{\partial X^i}{\partial x^k} \frac{\partial x^{\beta_1}}{\partial X^{\alpha_1}} \frac{\partial x^{\beta_2}}{\partial X^{\alpha_2}} \cdots \frac{\partial x^{\beta_q}}{\partial X^{\alpha_q}} V^k_{x\beta_1\beta_2\cdots\beta_q}. \tag{9}$$

Using the complex coordinate stretching[2,3,8)] given by $X^i = \int^{x^i} \tilde{s}_i(\tau)\, d\tau = \int^{x^i} \tilde{s}_{i\mathrm{R}}(\tau) + \mathrm{j}\tilde{s}_{i\mathrm{I}}(\tau) d\tau$ with the two real functions $\tilde{s}_{i\mathrm{R}}(\tau)$ and $\tilde{s}_{i\mathrm{I}}(\tau)$, we have a relation:

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[1]khasegaw@mmm.muroran-it.ac.jp

$$V^{i}_{X\alpha_1\alpha_2\cdots\alpha_q} = V^{i}_{x\alpha_1\alpha_2\cdots\alpha_q}\tilde{s}_i(x^i) \times \left[\tilde{s}_{\alpha_1}(x^{\alpha_1})\tilde{s}_{\alpha_2}(x^{\alpha_2})\cdots\tilde{s}_{\alpha_q}(x^{\alpha_q})\right]^{-1}. \quad (10)$$

Here, j is the imaginary unit.

3. PMLs in the Orthogonal Coordinate Systems

Assuming that the same constitutive equations in the real coordinates exist in the complex coordinates, we have following relations:

$$\boldsymbol{P}^{\mathrm{c}} = \rho\partial\boldsymbol{u}^{\mathrm{c}}/\partial t, \quad (11)$$

$$T^{\mathrm{c}}_{ij} = C^{\mathrm{E}}_{ijkl}F^{\mathrm{c}}_{kl} - \underline{e_{ijk}}E^{\mathrm{c}}_{k}, \quad (12)$$

$$D^{\mathrm{c}}_{i} = e_{ikl}F^{\mathrm{c}}_{kl} + \overline{\varepsilon^{S}_{ik}}E^{\mathrm{c}}_{k}. \quad (13)$$

Here, the superscript c denotes the value in the complex coordinate and the mass density ρ, the stiffness C^{E}_{ijkl}, the piezoelectric stress constants $\underline{e_{ijk}}$, e_{ikl}, and the permittivity at constant strain $\overline{\varepsilon^{S}_{ik}}$ are the values corresponding to the original material parameters of its PML in the real coordinate. Using eq. (10) to eqs. (1)-(4) and eqs. (6) and (7), we obtain

$$P^{\mathrm{c}}_{i} = \frac{s_i}{s_0s_1s_2}P_i \quad \text{(no summation)}, \quad (14)$$

$$T^{\mathrm{c}}_{ij} = \frac{s_is_j}{s_0s_1s_2}T_{ij} \quad \text{(no summation)}, \quad (15)$$

$$F^{\mathrm{c}}_{ij} = \frac{s_i}{s_j}F_{ij} \quad \text{(no summation)}, \quad (16)$$

$$E^{\mathrm{c}}_{i} = \frac{1}{s_i}E_i \quad \text{(no summation)}, \quad (17)$$

$$D^{\mathrm{c}}_{i} = \frac{s_i}{s_0s_1s_2}D_i \quad \text{(no summation)}, \quad (18)$$

where $s_i = \tilde{s}_i h^{\mathrm{C}}_i/h^{\mathrm{R}}_i$. Here, h^{C}_i and h^{R}_i are the scale factors of the complex and the real orthogonal coordinates: for example, $h^{\mathrm{C}}_i = h^{\mathrm{R}}_i = 1$ (i = 0,1,2) in the Cartesian coordinates.[13)] The quotient rule and eqs. (12)-(18) yield PML material constants: the mass density ρ^{PML} and the stiffness C^{PML}_{ijkl} are

$$\rho^{\mathrm{PML}} = s_0s_1s_2\rho, \quad (19)$$

$$C^{\mathrm{PML}}_{ijkl} = \frac{s_0s_1s_2s_k}{s_is_js_l}C_{ijkl} \text{(no summation)}, \quad (20)$$

and the permittivity and the piezoelectric stress constants are

$$\varepsilon^{\mathrm{PML}}_{ik} = \frac{s_0s_1s_2}{s_is_k}\varepsilon_{ik} \quad \text{(no summation)}, \quad (21)$$

$$\underline{e^{\mathrm{PML}}_{ijk}} = \frac{s_0s_1s_2}{s_is_js_k}\underline{e_{ijk}} \quad \text{(no summation)}, \quad (22)$$

$$e^{\mathrm{PML}}_{ikl} = \frac{s_0s_1s_2s_k}{s_is_l}e_{ikl} \text{(no summation)}. \quad (23)$$

Eqs. (19) and (20) for anisotropic solids were presented in the previous papers.[12,13)] The piezoelectric stress constants $\underline{e^{\mathrm{PML}}_{ijk}}$ and e^{PML}_{ikl} lose the transpose symmetry relation. We note that the stress tensors are symmetric[12,13)]: $T^{\mathrm{c}}_{ij} = T^{\mathrm{c}}_{ji}$ for $i \neq j$.

4. Comparison with PML Constants Derived From Differential Forms and the Analytic Continuation in the Cartesian Coordinates

By the analytic continuation, the mass density and stiffness of PML are given as follows:

$$\rho^{\mathrm{PMLA}} = s_0s_1s_2\rho, \quad (24)$$

$$C^{\mathrm{PMLA}}_{ijkl} = \frac{s_0s_1s_2}{s_js_l}C_{ijkl} \quad \text{(no summation)}. \quad (25)$$

The permittivity and the piezoelectric stress constants are[9-11)]

$$\varepsilon^{\mathrm{PMLA}}_{ik} = \frac{s_0s_1s_2}{s_is_k}\varepsilon_{ik} \quad \text{(no summation)}, \quad (26)$$

$$\underline{e^{\mathrm{PMLA}}_{ijk}} = \frac{s_0s_1s_2}{s_js_k}\underline{e_{ijk}} \quad \text{(no summation)}, \quad (27)$$

$$e^{\mathrm{PMLA}}_{ikl} = \frac{s_0s_1s_2}{s_is_l}e_{ikl} \quad \text{(no summation)}. \quad (28)$$

Note that the transpose symmetry relation holds in the piezoelectric stress constants derived by analytic continuation. However, the stress tensors computed by C^{PMLA}_{ijkl} are not symmetric: $T^{\mathrm{c}}_{ij} \neq T^{\mathrm{c}}_{ji}$ for $i \neq j$.

The mass density and the permittivity, eqs. (24) and (26), are identical to our results, eqs. (19) and (21). The stiffness and the piezoelectric stress constants that are derived by analytic continuation, eqs. (25), (27) and (28), are different from eqs. (20), (22) and (23) because the coordinate transformation corresponding to the components of contravariant of rank 1 of the stress tensor and the displacement gradient, s_i in the eqs. (15) and (16), is excluded in the analytic continuation.

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