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Numerical Dispersion of Elastic Wave Propagation in Finite Difference Time Domain Analysis
有限差分時間領域解析における弾性波伝搬の数値分散

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1. Introduction

For modeling propagation of elastic waves in anisotropic solids by finite-difference time-domain (FD-TD) method, we presented a staggered grid with the collocated grid points of velocities (SGCV)[1)]. To impose boundary conditions on the FD-TD model simply, the new grid was derived from a single control volume of the momentum conservation law and line integrations of the displacement gradient. Abandoning the cross-shape arrangement of the velocity vector results in interpolations of the velocity components away from grid points. In an isotropic solid in two dimensions, numerical dispersion relations of vertically polarized shear waves (SV-waves) and longitudinal waves (P-waves) modeled by (2,2) and (2,4) schemes have been derived and investigated numerically.[1)]

Because the previous paper[1)] focused on the derivation of the SGCV, we reported one numerical example of the numerical dispersion in the isotropic solid with a Poisson ratio of 0.3 for a FD-TD model with a Courant number $R=V_p\Delta_t/\Delta$ of 0.5 and a normalized spatial interval Δ/λ_s of 0.1 where V_p, Δ_t, Δ and λ_s are the phase velocity of the P-wave propagating in the solid, the time and the spatial intervals, and the wavelength of the SV-wave used in the analysis. We concluded that the interpolation with 3rd degree bi-polynomials gives comparable results of conventional staggered grids.[2,3)] Recalling that the numerical dispersion relations of the SGCV depend on the Poisson ratio but those of the conventional staggered grids do not, we should investigate the numerical dispersions for other Poisson ratios.

In this paper, we will present computed results of numerical dispersions of SV- and P-waves propagating in infinite isotropic solids with the Poisson ratio in the range of 0.1 to 0.495 by the FD-TD models with the (2,2) and (2,4) schemes in two dimensions. These results will show the usefullness of the SGCV models with the Poisson ratio in the range of 0.1 to 0.45.

2. FD-TD Models with the SGCV in Two Dimensions

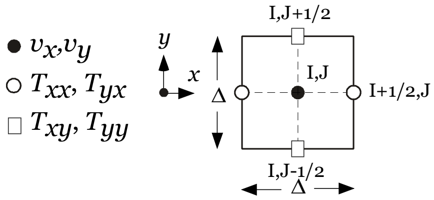

Fig.1 A unit cell of staggered grid with collocated grid points of velocities in two dimensions for P- and SV-waves propagation.

In an isotropic solid in two dimensions, the uniform SGCV for P- and SV-waves propagation reduces to a grid shown in **Fig. 1**. Here, I and J are integers for a grid point with the position vector $\boldsymbol{p} = (\mathrm{I}\hat{x} + \mathrm{J}\hat{y})\Delta$ where $\hat{x}$, $\hat{y}$ are the unit vectors in the directions of x- and y-axis, and v_i and T_{ij} ($i,j = x, y$) are the i-component of a particle velocity and the ij-component of a stress tensor, respectively.

Newton's equation of motion and relations of displacement gradient tensors Γ_{ij} and velocity vectors are modeled as follows:

$$\rho \mathrm{D}^T_{\mathbf{p},t}[v_i] = \mathrm{D}^T_{\mathbf{p},x}[T_{ix}] + \mathrm{D}^T_{\mathbf{p},y}[T_{iy}] \qquad \text{for } i=x,y, \quad (1)$$

$$\partial\Gamma_{ij}/\partial t\Big|_{\mathbf{p}}^{T+\Delta_t/2} = \mathrm{D}^{T+\Delta_t/2}_{\mathbf{p},j}[v_i] \qquad \text{for } i,j=x,y. \quad (2)$$

Here, ρ is the mass density, t is time, and $\mathrm{D}^T_{\mathbf{p},i}[f]$ is a finite difference approximation of the spatial ($i = x, y$) or time ($i = t$) derivative of a scalar function $f(\boldsymbol{r},t)$ with respect to i on the grid point $\mathbf{p}$ (● for eq. (1), □ and ○ for eq. (2)) where $T = K\Delta_t$ with an integer K. Using the derivative of the stress and strain relation with respect to time and eq. (2), we obtain following relation:

$$D^{T+\Delta_t/2}_{\mathbf{p},t}[T_{ij}] = \sum_{k,l} \left(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk}\right) D^{T+\frac{\Delta_t}{2}}_{\mathbf{p},l}[v_k]. \quad (3)$$

Here λ and μ are the Lamé constants, and δ_{ij} is the Kronecker delta.

Velocity gradients $D^{T+\Delta_t/2}_{\mathbf{p},y}[v_k]$ and $D^{T+\Delta_t/2}_{\mathbf{p},x}[v_k]$ for $k = x, y$ on the grid points of the stress components T_{kx} and T_{ky}, respectively, are required for the time update eq. (3). We used interpolations of velocity vectors on the four

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corners by a tensor product of two polynomial interpolations on adjoining grids[1] with the $(D_x \times D_y)$ values on the D_x and D_y grids in the x- and y-directions, respectively, as follows:

$$v_k(x,y) = \sum_{l=0}^{D_x-1} \sum_{m=0}^{D_y-1} C_{lm}^{k} x^l y^m. \quad (4)$$

3. Numerical Dispersion Relations of Plane SV- and P-waves in an Isotropic Solid

We consider monochromatic elastic plane waves propagating in an infinite solid with wave vector $\boldsymbol{k}$ on the x-y plane and the angular frequency ω. Assuming the dependence of particle velocities on discretized time and spatial grid points as exp j$(\omega K\Delta_t$-$\boldsymbol{k}\cdot\boldsymbol{p})$ and deriving wave equations from eqs. (1) and (3), we obtain the dispersion relation:

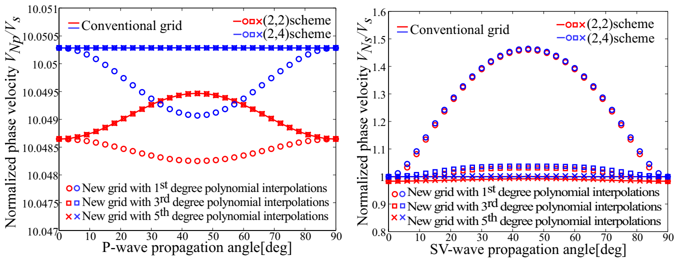
$$\sin^2\left(\omega\frac{\Delta t}{2}\right) = R^2\left(1-\frac{1}{4-4\sigma}\right)\left[\left(\widetilde{S_x^2}+\widetilde{S_y^2}\right) \pm \frac{1}{3-4\sigma}\sqrt{\left(\widetilde{S_x^2}+\widetilde{S_y^2}\right)^2 - 4\widetilde{S_x^2}\widetilde{S_y^2}(1-C_{\mathrm{D}})}\right], \quad (5)$$

where $R = V_p\frac{\Delta t}{\Delta}$, σ is the Poisson ratio, $\widetilde{S_\iota} = \sin\left(\boldsymbol{k}\cdot\frac{\hat{\iota}\Delta}{2}\right)$ (l = x, y) for (2,2) scheme and $\widetilde{S_\iota} = \sin\left(\boldsymbol{k}\cdot\frac{\hat{\iota}\Delta}{2}\right)\left[1+\frac{\sin^2\left(\boldsymbol{k}\cdot\frac{\hat{\iota}\Delta}{2}\right)}{6}\right]$ for (2,4) scheme, the sign + and − are for the P- and SV-waves in the double sign ±, and C_{D} is 1, $(1-S_x^2)(1-S_y^2)$, $(1-S_x^2)(1-S_y^2)(1+\frac{S_x^2}{2})^2(1+\frac{S_y^2}{2})^2$, $(1-S_x^2)(1-S_y^2)(1+\frac{S_x^2}{2}+\frac{3S_x^4}{8})^2(1+\frac{S_y^2}{2}+\frac{3S_y^4}{8})^2$ for the conventional staggered grids[2,3] and the SGCV with $D_x = D_y = 1, 3, 5$, respectively.

4. Computed Results of Numerical Dispersions

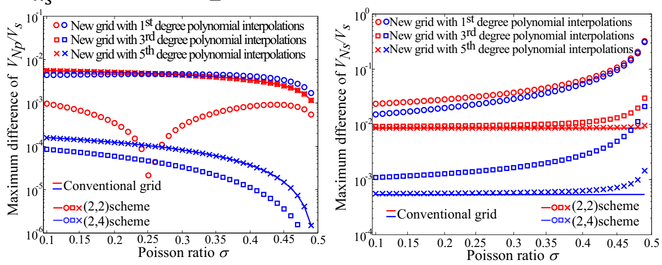
Figure 2 shows the computed results for the numerical dispersions with σ= 0.495, $\frac{\Delta}{\lambda_s} = 0.1$ and $R = 0.5$, where $\lambda_s = 2\pi V_s/\omega$ and $V_s = \sqrt{\mu/\rho}$. Here, V_{Np} and V_{Ns} are the values of P- and SV-wave velocities given by eq. (5) and the propagation angle of plane waves θ is defined as $\theta = \cos^{-1}(\boldsymbol{k}\cdot\hat{x}/|\boldsymbol{k}|)$. The values V_{Np} and V_{Ns} of the SGCVs are the same as those of conventional grids for θ= 0, π/2 and the maximum difference between the values appeare for θ= π/4. These results have been confirmed with σ= 0.3.[1] In the following results, we will show the absolute value of the difference between computed results of the staggered grid with θ= π/4 and θ= 0 for the maximum numerical dispersion on the grid.

Figure 3 shows the maximum difference of computed numerical dispersions as a function of σ in the range of 0.1 to 0.495. Interpolation with $D_x = D_y$= 5 for the SGCVs except SV-waves in solids



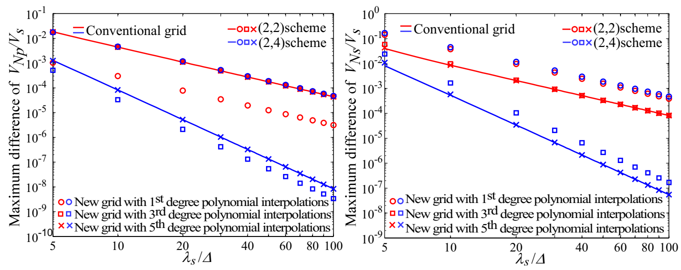
(a) P-waves (b) SV-waves

Fig.2 Numerical dispersions of (a) P- and (b) SV-waves propagation in an infinite isotropic solid with σ= 0.495, $\frac{\Delta}{\lambda_s} = 0.1$ and $V_p\frac{\Delta t}{\Delta} = 0.5$.



(a) P-waves (b) SV-waves

Fig.3 Maximum numerical dispersions of (a) P- and (b) SV-waves propagation in an infinite isotropic solid as functions of σ with $\frac{\Delta}{\lambda_s} = 0.1$ and $V_p\frac{\Delta t}{\Delta} = 0.5$.



(a) P-waves (b) SV-waves

Fig.4 Maximum numerical dispersions of (a) P- and (b) SV-waves propagation in an infinite isotropic solid as functions of $N = \lambda_s/\Delta$ with σ = 0.3 and $V_p\frac{\Delta t}{\Delta} = 0.5$.

with the Poisson ratio over 0.45 by (2,4) scheme reduces the numerical dispersions to the levels of the conventional grids.

Figure 4 shows the maximum difference of computed numerical dispersions as a function of $N = \lambda_s/\Delta$ with σ= 0.3 and $R = 0.5$. With increasing N, numerical dispersions of the P- and SV-waves decrease.

We conclude that the SGCV models of the SV-waves propagation in the solids with a large Poisson ratio such as 0.49 must be divided in finer grids than the conventional staggered grids. However, the SGCV models of P-waves require coarse grids comparable to the conventional grids.

References

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