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Numerical analysis for acoustic resonance of St.Venant-Kirchhoff hyperelastic sheet

二次元 St.Venant-Kirchhoff 型超弾性体の共鳴振動解析

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1. Introduction

It is well known that elastic constants C_{ijkl} plays an important role in materials science and condensed matter physics. To the best of the authors knowledge, resonant ultrasound spectroscopy (RUS)[1,2] is the state of the art technique because (i) it determines a complete set of C_{ijkl} tensor from one single crystal sample and (ii) measurement accuracy is sufficiently high. Because of these reasons that the RUS method has been applied to various kinds of materials. However, since the theory of RUS is written within a framework of the linear elasticity, it cannot be applied to a measurement of higher-order elastic constants. In order to determine not only the second elastic constants but also those of higher-orders from resonant frequency of an elastic medium, we must extend the theory into a general framework of nonlinear elasticity. In this study, we developed a theory that describes free-vibration acoustic resonance of two-dimensional *nonlinear* hyperelastic medium.

2. Theory

2.1. Variational Formulation

Let us consider a St.Venant-Kirchhoff type hyperelastic medium defined by $\Omega = \{x_i | -L_i < x_i < L_i, i = 1,2\}$. Let $u_i = u_i(t, x_i)$ be displacement functions defined in Ω. Then, the action integral I of the elastic medium is defined by the following form.

$$I[u_i] = \int_{-\pi/\omega}^{\pi/\omega} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} \mathcal{L}(t, x_1, x_2)\, dx_1 dx_2 dt. \qquad (1)$$

Here ω is a resonant frequency and $\mathcal{L}$ is the Lagrangian density. Since the domain Ω is consisted of the St.Venant-Kirchhoff hyperelastic medium, the Lagrangian density can be expressed by the following form.

$$\mathcal{L} = \frac{1}{2}\rho\left({u_{1,t}}^2 + {u_{2,t}}^2\right) - \frac{1}{2}\lambda(\mathrm{tr}\boldsymbol{E})^2 - \mu\,\mathrm{tr}(\boldsymbol{E}^2). \qquad (2)$$

Where $u_{i,t}$ is displacement velocity (material time derivative of u_i), $\boldsymbol{E}$ is Green-Lagrange strain tensor, and λ and μ are Lame constants, respectively. Since the elastic medium Ω is holonomic and free from any dissipation, the principle of stationary action must hold. In other words, a resonant state can be understood as a stationary point $\delta I = 0$. This is a variational problem defined on a variable domain with respect to time *t*.

2.2. Numerical analysis by Ritz method

The variational problem has been solved numerically by the Ritz method. First, we expand the deformation function by complex Fourier series in such a way that

$$u_i(t, x_1, x_2) = \sum_{n=-N_t}^{N_t} \sum_{p=1}^{P} F_{i,n,p}(x_1, x_2) e^{jn\omega t}. \qquad (3)$$

Here $F_{i,n,p} = \alpha_{i,n,p}\phi_p(x_1, x_2)$ consists of the four types of the basis functions:

$$F_{1,n,p} = a_{n,p}\phi_p + b_{n,p}\varphi_p + c_{n,p}\chi_p + d_{n,p}\psi_p,$$
$$F_{2,n,p} = e_{n,p}\phi_p + f_{n,p}\varphi_p + g_{n,p}\chi_p + h_{n,p}\psi_p. \qquad (5)$$

where

$$\phi_p = \frac{1}{\sqrt{L_1 L_2}} \sum_{m=0}^{M} \sum_{k=0}^{K} \sin\frac{(2m+1)\pi x_1}{2L_1} \sin\frac{(2k+1)\pi x_2}{2L_2},$$

$$\varphi_p = \frac{1}{\sqrt{L_1 L_2}} \sum_{m=0}^{M} \sum_{k=0}^{K} \sin\frac{(2m+1)\pi x_1}{2L_1} \cos\frac{k\pi x_2}{L_2},$$

$$\chi_p = \frac{1}{\sqrt{L_1 L_2}} \sum_{m=0}^{M} \sum_{k=0}^{K} \cos\frac{m\pi x_1}{L_1} \sin\frac{(2k+1)\pi x_2}{2L_2},$$

$$\psi_p = \frac{1}{\sqrt{L_1 L_2}} \sum_{m=0}^{M} \sum_{k=0}^{K} \cos\frac{m\pi x_1}{L_1} \cos\frac{k\pi x_2}{L_2}.$$

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Inserting Eqs. (2) and (3) into Eq. (1), we obtained analytic form of the action integral. Here, we introduce the L^2 norm of the displacement at $t = 0$:

$$\|u_i\|_{L^2} = \left[\int_{-L_2}^{L_2}\int_{-L_1}^{L_1}({u_1}^2 + {u_2}^2)\,dx_1dx_2\right]^{\frac{1}{2}}. \qquad (3)$$

Then, the variational problem can be understood as the minimization of strain energy under the constraint condition of $\|u_i\|_{L^2}$ = const. This problem ends up with a nonlinear simultaneous equation and is solved numerically through a convergence calculation by the Newton method. Note that we set $\rho = 1.0$, $L_1 = L_2 = 1.0$ and $\lambda = 3\mu = 1$. The order of the basis function is set to be $M = K = 3$: the degree of freedom is $X = 8 \times P = 8 \times (M+1)(K+1) = 128$.

3. Results and Discussion

3.1. Amplitude dependence of resonance frequency

Figure 1 shows amplitude dependence of resonance frequency ω obtained from the first resonant vibration mode. As seen from the figure, ω decreases monotonically with increasing in amplitude. This is due to the nonlinearity of the medium. At the low amplitude limit, $\|u_i\|_{L^2} \to 0$, resonance frequency ω converges to the linear case solution, as it should be. Similar features have been confirmed in other resonant vibration modes and these result consistent with the previous study for an one-dimensional nonlinear elastic bar model[3].

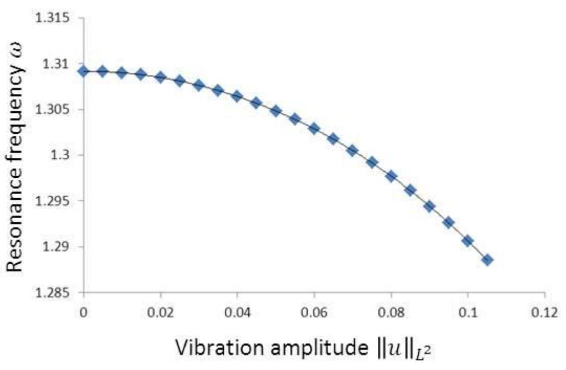

Figure 1. Amplitude dependence of resonance frequency obtained from the 1st resonant vibration mode.

3.2. Vibration patterns

Figure 2 shows resonant vibration patterns obtained from the first three resonant vibration modes at $t = 0$ (top) and $t = \pi/\omega$ (bottom). These patterns are similar to those of linear system. However, careful investigation revealed that (i) these vibration patterns include high wave number as well as high frequency modes, (ii) there exists no specific time at which all displacement vanish in Ω, and (iii) the vibration patterns can be classified into four vibration group.

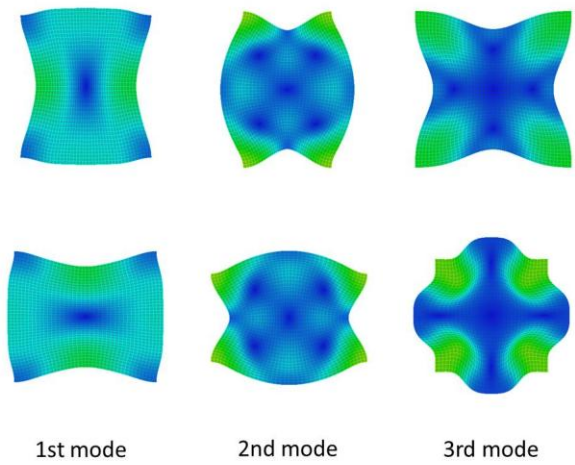

Figure 2. Resonant vibration patterns obtained from the first three resonant vibration modes.

4. Conclusions

Free vibration acoustic resonance of a two-dimensional St.Venant-Kirchhoff hyperelastic medium has been investigated using the theory of nonlinear elasticity and a direct analysis in the calculus of variation. Present study revealed the following properties of the elastic medium:

1. Resonance frequency ω of the hyperelastic medium shows marked amplitude dependence.
2. Both high frequency and high wavenumber modes are excited due to the nonlinearity of the medium.

References

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