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An Analysis of Lamé Mode Resonators by Finite Difference Time Domain Method

ラーメモード振動子の FD-TD 解析

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1. Introduction

A staggered grid with the collocated grid points of velocities (SGCV) was presented for modeling propagation of elastic waves in anisotropic solids by finite-difference time-domain (FD-TD) method[1)]. To impose boundary conditions on the FD-TD model simply, the new grid was derived from a single control volume of the momentum conservation law and line integrations of the displacement gradient. Abandoning the cross-shape arrangement of the velocity vector results in interpolations of the velocity components away from grid points. Numerical dispersions of vertically polarized shear waves (SV-waves) and longitudinal waves (P-waves) in an infinite isotropic solid by (2,2) and (2,4) schemes have been investigated and it has been reported[1)] that the interpolation with 3rd degree bi-polynominals gives comparable results of conventional staggered grids[2,3)].

For modeling elastic wave devices by FD-TD method, boundary conditions on planar free surfaces should be examined. With the conventional staggered grids[2,3)], FD-TD models can use the stress-imaging technique[3)], the vacuum formalism[4)] or the adjusted staggered scheme[5)]. Because the velocity vectors are on the center grid points and all stresses on the grid surfaces are normal components in the FD-TD models with the SGCV, imposing of planar free surface conditions is to set all stress components on the surface zero. However, the interpolations of the velocity vectors away from the grid points near the free surface are required to compute velocity gradients. Hence a modification must be introduced. Extrapolation techniques or one-sided finite differentiation schemes are cacandidates for immediate computation of the velocity gradients on the grid points of stresses.

In this paper, for computation of velocity gradients, we will examine the derivatives of interpolation polynomials of velocity vector fields for the FD-TD models with the SGCV in two dimensions. We will extract the resonance frequency of the fundamental Lamé mode of a resonator on an isotropic solid and confirm the validity of the SGCV models of the free surfaces.

2. Modeling Planar Free Surfaces by FD-TD with the SGCV in Two Dimensions

2.1 Time update equations by FD-TD scheme

In an isotropic solid in two dimensions, the cell of a uniform SGCV for SV- and P-waves propagation reduces to a grid shown in **Fig. 1**. Here, Δ is the spatial interval of the grid, I and J are integers for a grid point with the position vector $\boldsymbol{p} = (I\hat{x} + J\hat{y})\Delta$ where $\hat{x}$, $\hat{y}$ are the unit vector in the direction of x- and y-axis, and v_i and T_{ij} ($i, j = x, y$) are the i-component of a particle velocity and the ij-component of a stress tensor.

Newton's equation of motion and relations of displacement gradient tensors Γ_{ij} and velocity vectors are modeled by FD-TD schemes in two dimensions as follows:

$$\rho \mathrm{D}^{T}_{\mathbf{p},t}[v_i] = \mathrm{D}^{T}_{\mathbf{p},x}[T_{ix}] + \mathrm{D}^{T}_{\mathbf{p},y}[T_{iy}] \quad \text{for } i = x, y, \quad (1)$$

$$\partial \Gamma_{ij} / \partial t\Big|^{T+\Delta_t/2}_{\mathbf{p}} = \mathrm{D}^{T+\Delta_t/2}_{\mathbf{p},j}[v_i] \quad \text{for } i, j = x, y. \quad (2)$$

Here, ρ is the mass density, t is time, and $\mathrm{D}^{\mathrm{T}}_{\mathbf{p},i}[f]$ is a finite difference approximation of the spatial (i=x, y) or time (i=t) derivative of a scalar function $f(\boldsymbol{r},t)$ with respect to i on the grid point $\mathbf{p}$ (●:eq. (1), □ and ○:eq. (2)) where $T = K\Delta_{\mathrm{t}}$ with an integer K and the time interval Δ_{t}. Using the derivative of the stress and strain relation with respect to time and eq.(2), we obtain following relations:

$$D^{T+\Delta_t/2}_{\mathbf{p},t}[T_{ij}] = \sum_{k,l} C_{ijkl} D^{T+\Delta_t/2}_{\mathbf{p},l}[v_k] \text{ for } i, j = x, y. \quad (3)$$

The stiffness component C_{ijkl} of the isotropic solid is

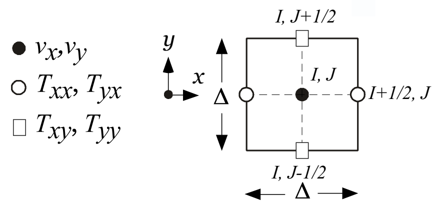

Fig.1 A unit cell of staggered grid with collocated grid points of velocities in two dimensions for SV- and P-waves propagation.

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given by $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Here λ and μ are the Lamé constants, and δ_{ij} is the Kronecker delta.

Velocity gradients $D_{\mathbf{p},y}^{T+\Delta_t/2}[v_k]$ and $D_{\mathbf{p},x}^{T+\Delta_t/2}[v_k]$ for k=x,y on the grid points of the stress components T_{kx} and T_{ky}, respectively, are required for the time update eq. (3). Using the finite difference approximation, therefore, velocity vectors on the four corners of the SGCV grid were interpolated in the elastic solid[1].

2.2 FD-TD model of free surfaces by SGCV

Consider an edge of a SGCV grid on a free surface. Velocity gradients $D_{\mathbf{p},y}^{T+\Delta_t/2}[v_k]$ or $D_{\mathbf{p},x}^{T+\Delta_t/2}[v_k]$ (k = x, y) on the center point of the edge whose normal unit vector is $\hat{x}$ or $\hat{y}$, respectively, cannot be computed by the interpolation scheme in the infinite solid[1] because of lack of the grid points of the velocity vector in the vacuum. Recalling that the derivative of an interpolation polynomial is a scheme for numerical differentiation, we can compute the required gradients $v_{k,x}$ or $v_{k,y}$ on the edge immediately: when a tensor product of two polynomial interpolations of k-component of the velocity vector on adjoining grids, $v_k(x,y) = \sum_{l=0}^{D_x-1}\sum_{m=0}^{D_y-1} C_{lm}(x-x_0)^l(y-y_0)^m$ for k=x, y, is used, we obtain $v_{k,x}(x_0,y_0) = C_{10}$ or $v_{k,y}(x_0,y_0) = C_{01}$. Here, the coefficient C_{10} or C_{01} is computed by the $(D_x \times D_y)$ values on the D_x and D_y adjoining grids in the x- and y-directions, respectively.

3. Analysis of a Lamé Mode Resonator

3.1 FD-TD models

In two dimensions, we consider a Lamé mode resonator that is a square with a side length of L on an isotropic solid with Poisson's ratio 0.25. When the wavelength of the SV-wave at the frequency f_s is $2L$, the fundamental resonance frequency f_1 of the Lamé mode is $f_1 = \sqrt{2}f_s$. In the following results, f_1 = 1MHz and $R = \frac{v_p\Delta_t}{\Delta} = 0.5$, where v_p is the phase velocity of the P-wave in the infinite solid. Near the free surface, for computation of $v_{k,x}(x_0,y_0)$ or $v_{k,y}(x_0,y_0)$, we choose $D_x = D_y$ =2 or D_j = 3 and the other of 2. In the solid away from the surfaces, we use a bilinear polynomial interpolation with four nodes for computation of $D_{\mathbf{p},y}^{T+\Delta_t/2}[v_k]$ and $D_{\mathbf{p},x}^{T+\Delta_t/2}[v_k]$.

3.2 Computation of the resonance frequency

The observation point and vibration point are (L/4, L/4) and (-L/4, -L/4) on the x-y plane with the origin on the center of the square resonator. The vibration of the x-component of the particle velocity expressed as a sine-modulated Gaussian pulse with the center frequency f_1 and the half-width in time $83.26RL/v_p$ yields a discrete time response of the particle velocity of the observation point as shown in **Fig. 2**. Applying the discrete Fourier transform to the time response at the observation point in an interval from $N_s\Delta_t$ to $N_e\Delta_t$, we extract the resonance frequency of the resonator. **Figure 3** shows the power spectrum with $L/h = 2^6$, $N_s\Delta_t = 2^8\,RL/v_p \approx 52.26$μs, $N_e\Delta_t = 2^{16}\,RL/v_p$ and the coefficients, C_{10} or C_{01}, computed by the six adjoining grids. Extracted resonance frequency is 0.9999MHz and we may confirm the validity of our free surface models.

Figure 4 shows extracted resonance frequencies with $N_s\Delta_t = 2^8RL/v_p \approx 52.26$μs and $N_e\Delta_t = 2^N RL/v_p$. Computed results by four and six adjoining grids with L/h = 4 are 0.992 and 0.985MHz, respectively, and we expect that use of six adjoining grids is preferable for an accurate model.

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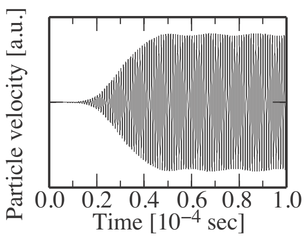

Fig.2 A time response at the observation point.

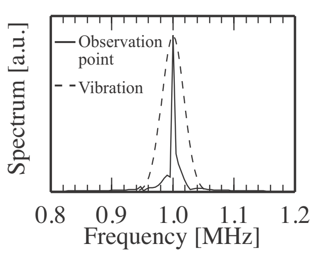

Fig.3 Power spectra of x-components of particle velocities of the vibration and observation points.

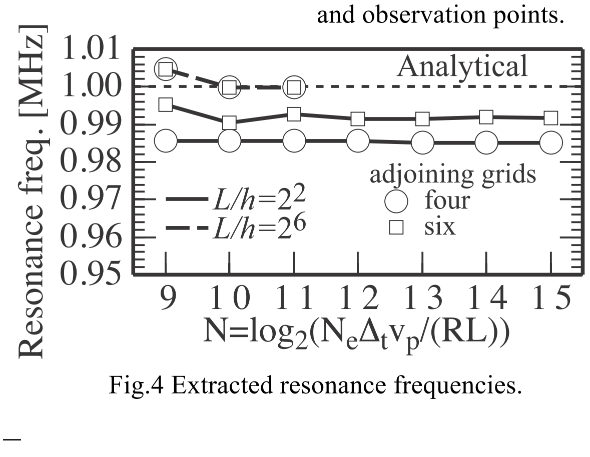

Fig.4 Extracted resonance frequencies.