2P1-7

Proceedings of Symposium on Ultrasonic Electronics, Vol. 34 (2013) pp. 201-202
20-22 November, 2013

Free-vibration Acoustic Resonance of Two-dimensional Hyperelastic Material
非線形超弾性体の共鳴振動解析

Ryuichi Tarumi[1†], Shinpei Yamada[1], and Yoji Shibutani[1] ([1]Dep. of Mechanical Engineering, Osaka Univ.)
垂水竜一[1†]，山田晋平[2]，渋谷陽二[1]([1]阪大・工，[2]阪大・院)

1. Introduction

Free-vibration acoustic resonance (FVAR) is a characteristic feature of solids as it expresses their eigen vibration. Since the pioneering work of Rayleith and Ritz, FVAR of solid has been investigated extensively both from scientific and engineering viewpoint. However, most of the previous works have been conducted for linear elastic medium and FVAR of nonlinear solid is still relatively little understood. In the present study, we investigated FVAR of a two-dimensional nonlinear hyperelastic material for the first time. Our formulation is based on the principle of stationary action, theory of nonlinear elasticity and calculus of variation. Numerical analysis, based on the Ritz method, demonstrated that *colour symmetry* is embedded in the finite amplitude nonlinear FVAR modes. More precisely, we revealed the existence of four types of noninear FVAR modes that can be classified on the basis of a *single colour* (black or wthite) and three types of *bicolour* (black and white) magnetic point groups, rather the irreducible representations used for a linearized system. We also predicted a *grey colour* nonlinear FVAR mode which lacks the harmonic vibration component.

2. Theory of nonlinear FVAR

2.1. Variational Formulation

Let $\Omega=\{(x_1,x_2)|-L_i<x_i<L_i, i=1,2\}$ be a two-dimensinal St. Venant- Kirchhoff noninear hyperelastic domain and let $u_i=u_i(x_1,x_2,t)$ be a displacement due to FVAR. Note that the displacement function is expressed in the Lagrange description and is continuously differentiable with respect to the all arguments. Then, the kinetic energy density and nonlinear strain energy density of Ω are given by

$$\mathcal{K}=\frac{\rho}{2}\left(u_{1,t}^2+u_{2,t}^2\right),\quad \mathcal{W}=\frac{\lambda}{2}E_{ii}^2+\mu E_{ij}E_{ji},$$

where ρ is the mass-density and λ and μ are Lame constants. E_{ij} is the Green-Lagrange strain tensor defined on the reference configuration Ω. Hereafter, summation convention holds for repeated indices. Integration of the Lagrangian density, which is defined by $\mathcal{L}=\mathcal{K}-\mathcal{W}$, over the fixed domain $x_i\in\Omega$ and a *variable* time interval $t\in(0,2\pi/\omega)$ yields the action integral such that

$$I[u_i]=\int_0^{2\pi/\omega}\int_\Omega \mathcal{L}(u_{i,t},\nabla u_i)\,dV\,dt.$$

Because the hyperelastic material Ω is a holonomic system and free from any dissipation, principle of the stationary action hold. It states that, among the continuously differentiable functions defined on $x_i\in\Omega$ and $t\in(0,2\pi/\omega)$, the actual displacement must satisfy the stationary condition $\delta I=0$. The corresponding Euler-Lagrange equation (in strong form) and boundary conditions reveal that the stationary solution should satisfy the time periodicity and time reversal symmetry conditions. However, they are insufficient to well-pose the initial-boundary value type nonlinear partial differential equation. We therefore solve the variational problem by direct method.

2.2. Numerical analysis by the Ritz method

According to the time periodicity and time reversal symmetry, we expand the displacement function by the orthonormal Fourier series in such a way that

$$u_1=\sum_{n=0}^{N}\sum_{m=1}^{M}\left(a_{n,m}\phi_m+b_{n,m}\varphi_m+c_{n,m}\chi_m+d_{n,m}\psi_m\right)\cos(n\omega t),$$

$$u_2=\sum_{n=0}^{N}\sum_{m=1}^{M}\left(e_{n,m}\phi_m+f_{n,m}\varphi_m+g_{n,m}\chi_m+h_{n,m}\psi_m\right)\cos(n\omega t),$$

where

$$\phi_m=A_{m_1,m_2}\sin\frac{(2m_1+1)\pi x_1}{2L_1}\sin\frac{(2m_2+1)\pi x_2}{2L_2},$$

$$\varphi_m=A_{m_1,m_2}\sin\frac{(2m_1+1)\pi x_1}{2L_1}\cos\frac{m_2\pi x_2}{L_2},$$

$$\chi_m=A_{m_1,m_2}\cos\frac{m_1\pi x_1}{L_1}\sin\frac{(2m_2+1)\pi x_2}{2L_2},$$

$$\psi_m=A_{m_1,m_2}\cos\frac{m_1\pi x_1}{L_1}\cos\frac{m_2\pi x_2}{L_2}.$$

We also impose a subsidiary condition

†tarumi@mech.eng.osaka-u.ac.jp

$$\|u_i\|_{L^2} = \left[\int_\Omega (u_1^2 + u_2^2) dV \Big|_{t=0}\right]^{1/2} = \text{const.}$$

to compensate for a missing equation. For numerical analysis, we employed the dimensionless quantities $\rho = 1$, $L_1 = 1.1$, $L_2 = 0.9$, $\lambda = 1$ and $\mu = 1/3$ for simplicity. The order of the basis function is $N = 3$ and $m_1 = m_2 = 6$, therefore the degree of freedom is 1,566. Nonlinear equations are solved around a linearized solution by convergent calculation by the Newton method.

3. Results and discussion

Figure 1(a) shows a nonlinear FVAR pattern obtained from the initial data $A_1' - 2$, which has the second-lowest FVAR frequency in linearized A_1 mode. **Figure 2(b) to (e)** show $n = 0$ to $n = 3$ components in (a), thereby their summation yields (a). It is obvious from the figure that the vibration pattern (a) have the A_1 symmetry (total symmetry) as its components (b) to (e) have the A_1 in the entire vibration period. **Figure 2(a)** shows a nonlinear FVAR pattern obtained from the initial data $A_1' - 2$. Similar to **Fig. 1**, displacement components of (a) are shown in (b) to (e). As seen here, the principal harmonic component (c) and the third harmonics (e) has A_2 symmetry. However, unlike the previous case, symmetry of the second harmonics (d) and the time-independent component (b) is not A_2 but A_1. Similar features have been confirmed in other A_2' modes as well as B_1' and B_2' modes.

To explain the exotic vibration symmetry, we employ the magnetic point group, rather than the irreducible representations that have been used for classification of linearized FVAR modes. According to group theory, symmetry operations of a two-dimensional rectangle Ω form point group $C_{2v} = \{E, C_2, \sigma_x, \sigma_y\}$. Let T be the time-reversal operator. From the coset decomposition of C_{2v} of index 2, there exist five types of magnetic point groups:

1. $C_{2v}(C_{2v}) = \{E, C_2, \sigma_x, \sigma_y\}$,
2. $C_{2v}(C_2) = \{E, C_2\} + T \circ \{\sigma_x, \sigma_y\}$,
3. $C_{2v}(C_{1h}) = \{E, C_2\} + T \circ \{\sigma_x, \sigma_y\}$,
4. $C_{2v}(C_{1h}') = \{E, C_2\} + T \circ \{\sigma_x, \sigma_y\}$,
5. $C_{2v}(C_2) = \{E, C_2, \sigma_x, \sigma_y\} + T \circ \{E, C_2, \sigma_x, \sigma_y\}$.

The first one is *single colour* whereas the remaining four are *bicolour* magnetic point groups. From projection operations, we confirmed that the magnetic point groups 1 to 4 are responsible for vibration symmetry of A_1', A_2', B_1' and B_2'. This demonstrates that the *colour symmetry* is embedded in the nonlinear FVAR modes.

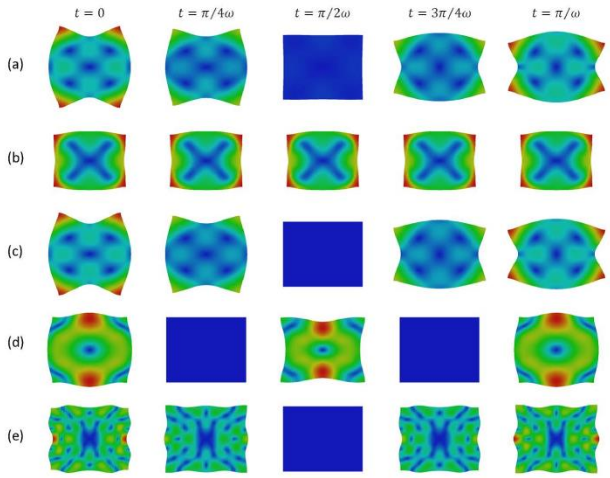

Figure 1. (a) FVAR pattern of $A_1' - 2$mode obtained at $\|u_i\|_{L^2} = 0.05$. (b) to (e) Displacement components for $n = 0$ to $n = 3$.

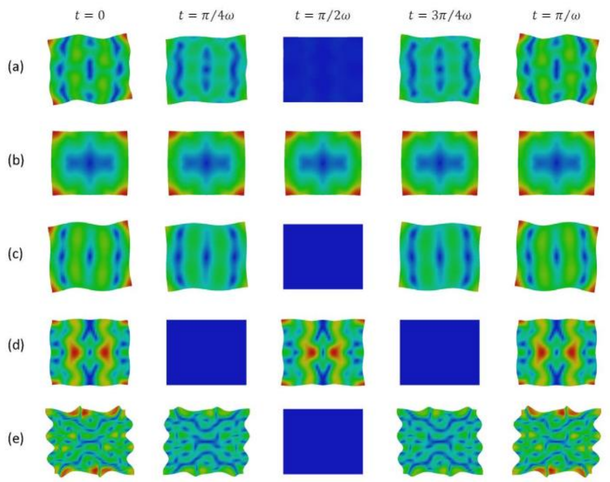

Figure 2. (a) FVAR pattern of $A_2' - 2$mode obtained at $\|u_i\|_{L^2} = 0.05$. (b) to (e) Displacement components for $n = 0$ to $n = 3$.

4. Conclusions

We investigated FVAR of a two-dimensional nonlinear hyperelastic material numerically by the Ritz method. Present results demonstrated that *colour symmetry* is embedded in the nonlinear FVAR. Therefore, magnetic point group, rather than the irreducible representations, is responsible for the symmetry and structure of the nonlinear FVAR modes.

References

1. L. Rayleigh: The theory of sound vol. 1 and 2 (Dover publications, New York 1945).
2. W. Ritz: J. die Reine & Angewandte Mathematik **135**, 1-61; Annalen der Physik **28**, 737-786.
4. R. Tarumi: Proc. Royal Society A (in press).